Simplifying Eigenvector Radiosity
(Updated)

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Making the simple complicated is commonplace; making the complicated simple, awesomely simple, that's creativity.

Charles Mingus, 1922 – 1979

Suppose we are given a system of linear equations in $\Re^n$ such as:

$$\mathbf{b} = \mathbf{Qx} \tag{1}$$

where $\mathbf{x}$ is an unknown $n \times 1$ vector, $\mathbf{Q}$ is a symmetric and nonsingular $n \times n$ matrix, and $\mathbf{b}$ is a known $n \times 1$ vector. We can rewrite this as:

$$\mathbf{x} = \mathbf{Q}^{-1}\mathbf{b} \tag{2}$$

The matrix $\mathbf{Q}$ may be expressed in terms of its eigenvectors $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and their associated eigenvalues $\mathbf{D} = diag(\lambda_1, \ldots, \lambda_n)$ as:

$$\mathbf{Q} = \mathbf{VDV}^T \tag{3}$$

and its inverse as:

$$\mathbf{Q}^{-1} = \mathbf{VD}^{-1}\mathbf{V}^T \tag{4}$$

so that:

$$\mathbf{x} = \mathbf{VD}^{-1}\mathbf{V}^T\mathbf{b} \tag{5}$$

As an example, suppose we are given the radiative flux transfer (aka radiosity) equation:

$$\mathbf{M}_o = (\mathbf{I} - \mathbf{RF})\mathbf{M} \tag{6}$$

where $\mathbf{M}$ is the final radiant exitance vector, $\mathbf{M}_o$ is the initial radiant exitance vector, $\mathbf{R}$ is the diagonal reflectance vector (where $r_{ii} = \rho_i$ and $r_{ij} = 0, i \neq j$), and $\mathbf{F}$ is the form factor matrix. Rearranging this equation, we have:

$$\mathbf{M} = (\mathbf{I} - \mathbf{RF})^{-1}\mathbf{M}_o \tag{7}$$

The matrix $(\mathbf{I} - \mathbf{RF})$ is asymmetric. However, the matrix $\mathbf{AR}^{-1}(\mathbf{I} - \mathbf{RF})$ is symmetric, where $\mathbf{A}$ is the diagonal matrix of surface areas[1]. This gives:

$$\mathbf{M} = (\mathbf{I} - \mathbf{RF})^{-1}\mathbf{M}_o = \left((\mathbf{I} - \mathbf{RF})^T\mathbf{AR}^{-1}\right)^{-1}\mathbf{AR}^{-1}\mathbf{M}_o = \mathbf{VD}^{-1}\mathbf{V}^T\mathbf{AR}^{-1}\mathbf{M}_o \tag{8}$$

[1] Ashdown, I. 2002. “Radiative Transfer Networks Revisited,” *Journal of the illuminating Engineering Society* 31(2):38-51 (Summer).

Given any symmetric $n \times n$ matrix $\mathbf{Q}$, the spectral decomposition theorem states that:

$$\mathbf{Q} = \mathbf{VDV}^T = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i^T \tag{9}$$

In many cases, the matrix $\mathbf{Q}$ can be approximated by a subset of the eigenvectors with the largest absolute magnitude eigenvalues. That is, if the eigenvectors are ordered such that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$, then:

$$\mathbf{Q} \approx \mathbf{Q}' = \sum_{i=1}^{p} \lambda_i \mathbf{v}_i \mathbf{v}_i^T \ , \ \ p << n \tag{10}$$

where $\mathbf{Q}'$ is rank deficient.

Applying this to equation 8, we have:

$$\mathbf{M} \approx \sum_{i=1}^{p} \lambda_i \mathbf{v}_i \mathbf{v}_i^T \mathbf{A}\mathbf{R}^{-1} \mathbf{M}_o \ , \ \ p << n \tag{11}$$

where λ_i and $\mathbf{v}_i$ are the i^{th} eigenvalue and eigenvector of $(\mathbf{I} - \mathbf{RF})^T \mathbf{AR}^{-1}$.

For $n < 400$ or so, the eigensystem of a symmetric matrix can be determined with the QR algorithm. For large matrices, the block Lanczos algorithm is appropriate to determine the eigenvectors with the largest absolute magnitude eigenvalues.

To conclude, equation 11 offers a direct solution to the radiosity equation 7. While the need to calculate the eigensystem may make this approach slower than Southwell iteration (aka *progressive radiosity*[2]), it is useful where the geometry remains fixed but the initial radiant exitance vector $\mathbf{M}_o$ is constantly changing.

[2] Ashdown, I. 1994. *Radiosity: A Programmer's Perspective.* New York, NY: John Wiley & Sons.